

# HEAT CONTENT ESTIMATES OVER SETS OF FINITE PERIMETER.

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**ABSTRACT.** This paper studies by means of standard analytic tools the small time behavior of the heat content over a bounded Lebesgue measurable set of finite perimeter by working with the set covariance function and by imposing conditions on the heat kernels. Applications concerning the heat kernels of rotational invariant  $\alpha$ -stable processes are given.

**Keywords:** covariance function, heat content, functions of bounded variation, stable processes, sets of finite perimeter.

## 1. INTRODUCTION

Let  $I$  be a set of indices and  $d \geq 2$  an integer. Consider a set of non-negative functions

$$\left\{ p_t^{(\alpha)}(\cdot) : \mathbb{R}^d \rightarrow [0, \infty], \alpha \in I, t \geq 0 \right\},$$

where each  $p_t^{(\alpha)}(\cdot)$  will be called *heat kernel*. We shall assume that these heat kernels satisfy the following properties.

- (i) For each  $t > 0$ ,  $p_t^{(\alpha)}(x)$  is radial. That is,  $p_t^{(\alpha)}(x) = p_t^{(\alpha)}(|x|) \geq 0$ ,  $x \in \mathbb{R}^d$ . Furthermore, we assume  $p_t^{(\alpha)}(\cdot) \in L^1(\mathbb{R}^d)$ .
- (ii) Scaling Property: for each integer  $d \geq 2$  and  $\alpha \in I$ , there exist  $\beta = \beta(d, \alpha) \in \mathbb{R}$  and  $\gamma = \gamma(d, \alpha) > 0$  such that

$$(1.1) \quad p_t^{(\alpha)}(x) = t^\beta p_1^{(\alpha)}(t^{-\gamma} x).$$

As a consequence of the aforementioned properties, we obtain

$$(1.2) \quad \begin{aligned} \|p_t^{(\alpha)}\|_{L^1(\mathbb{R}^d)} &= t^{\beta+d\gamma} \|p_1^{(\alpha)}\|_{L^1(\mathbb{R}^d)}, \\ p_t^{(\alpha)}(x) &= p_t^{(\alpha)}(|x| e_d), \end{aligned}$$

where  $e_d$  stands for the vector  $(0, 0, \dots, 0, 1) \in \mathbb{R}^d$ .

Before continuing, we provide some useful notations. Throughout the paper,  $\mathcal{L}(\mathbb{R}^d)$  will denote the set of all the Lebesgue measurable subsets of  $\mathbb{R}^d$ . For a bounded set  $\Omega \in \mathcal{L}(\mathbb{R}^d)$  with non-empty boundary  $\partial\Omega$ , we set

$$|\Omega| = \text{volume of } \Omega,$$

$$\mathcal{H}^{d-1}(\partial\Omega) = (d-1)\text{-Hausdorff measure of the boundary of } \Omega.$$

Henceforth,  $B_r(x)$  will stand for the ball centered at  $x \in \mathbb{R}^d$  with radius  $r$  and for simplicity  $B$  will represent the unit ball centered at zero. Also  $S^{d-1}$  will denote the boundary of the unit ball  $B$ . Moreover, the volume and surface area of the unit ball in  $\mathbb{R}^d$  will be denoted by  $w_d$  and  $A_d$ , respectively. That is,

$$(1.3) \quad \begin{aligned} w_d &= \frac{\pi^{\frac{d}{2}}}{\Gamma\left(1 + \frac{d}{2}\right)}, \\ A_d &= dw_d. \end{aligned}$$

In addition, if  $g : \Omega \subseteq \mathbb{R}^d \rightarrow \mathbb{R}$  is a Lipschitz function, we denote

$$\text{Lip}(g) = \sup \left\{ \frac{|g(y) - g(x)|}{|y - x|} : x, y \in \Omega, x \neq y \right\}.$$

Let  $\Omega \in \mathcal{L}(\mathbb{R}^d)$  be a bounded set. The purpose of the paper is to investigate the behavior as  $t \rightarrow 0+$  of the following function

$$(1.4) \quad \mathbb{H}_\Omega^{(\alpha)}(t) = \int_\Omega dx \int_\Omega dy p_t^{(\alpha)}(x - y),$$

which will be called *the heat content* of  $\Omega$  in  $\mathbb{R}^d$  by imposing conditions over the heat kernel  $p_t^{(\alpha)}(\cdot)$  and the underlying set  $\Omega$ . We remark that  $\mathbb{H}_\Omega^{(\alpha)}(t)$  is finite for all  $t > 0$  due to the assumption  $p_t^{(\alpha)}(\cdot) \in L^1(\mathbb{R}^d)$  and the inequality

$$0 \leq \mathbb{H}_\Omega^{(\alpha)}(t) \leq \int_\Omega dx \int_{\mathbb{R}^d} dy p_t^{(\alpha)}(x - y) = |\Omega| \|p_t^{(\alpha)}\|_{L^1(\mathbb{R}^d)}.$$

The function  $\mathbb{H}_\Omega^{(\alpha)}(t)$  turns out to provide information about the geometry of the set  $\Omega$  as long as regularity conditions over  $\Omega$  are assumed. For instance, in [20, Theorem 2.4] is proved by taking  $I = \{2\}$  and considering the Gaussian kernel

$$(1.5) \quad p_t^{(2)}(x) = (4\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{4t}\right)$$

that

$$\lim_{t \rightarrow 0+} \frac{|\Omega| - \mathbb{H}_\Omega^{(2)}(t)}{\sqrt{t}} = \frac{1}{\sqrt{\pi}} \mathcal{H}^{d-1}(\partial\Omega),$$

for  $\Omega$  a bounded domain with boundary being  $C^2$ . In [11], M. van den Berg called  $\mathbb{H}_\Omega^{(2)}(t)$  the heat content of  $\Omega$  in  $\mathbb{R}^d$  and therefore following the terminology introduced by M. van den Berg, we have also called  $\mathbb{H}_\Omega^{(\alpha)}(t)$  the heat content of  $\Omega$  in  $\mathbb{R}^d$ . We refer the interested reader to the papers [8, 9, 10] for recent results concerning bounds and asymptotic behaviors of the heat content corresponding to the case  $I = \{2\}$  and the Gaussian kernel over open sets, polygonal domains and its extensions when dealing with compact manifolds.

In order to investigate the small time behavior of  $\mathbb{H}_\Omega^{(\alpha)}(t)$ , we need to introduce the notion of finite perimeter. We say that a bounded set  $\Omega \in \mathcal{L}(\mathbb{R}^d)$  has finite perimeter if

$$(1.6) \quad 0 \leq \sup \left\{ \int_\Omega dx \operatorname{div} \varphi(x) : \varphi \in C_c^1(\mathbb{R}^d, \mathbb{R}^d), \|\varphi\|_\infty \leq 1 \right\} < \infty,$$

and we denote the last quantity by  $\text{Per}(\Omega)$ .

Our main result is the following.

**Theorem 1.1.** *Let  $d \geq 2$  be an integer. Consider  $\Omega \in \mathcal{L}(\mathbb{R}^d)$  a bounded set with  $\text{Per}(\Omega) < \infty$  and let  $w_{d-1}$  and  $A_d$  be the constants defined in (1.3).*

(i) *Let  $I_0 = \left\{ \alpha \in I : |\cdot| p_1^{(\alpha)}(\cdot) \in L^1(\mathbb{R}^d) \right\}$ . For each  $\alpha \in I_0$ , we have for all  $t > 0$  that*

$$(1.7) \quad \|p_1^{(\alpha)}\|_{L^1(\mathbb{R}^d)} |\Omega| - t^{-(\beta+d\gamma)} \mathbb{H}_\Omega^{(\alpha)}(t) \leq t^\gamma w_{d-1} \text{Per}(\Omega) \int_0^\infty dr r^d p_1^{(\alpha)}(r e_d).$$

*Furthermore,*

$$(1.8) \quad \lim_{t \rightarrow 0+} \frac{\|p_1^{(\alpha)}\|_{L^1(\mathbb{R}^d)} |\Omega| - t^{-(\beta+d\gamma)} \mathbb{H}_\Omega^{(\alpha)}(t)}{t^\gamma} = w_{d-1} \text{Per}(\Omega) \int_0^\infty dr r^d p_1^{(\alpha)}(r e_d).$$

(ii) Let

$$I_1 = \left\{ \alpha \in I : p_t^{(\alpha)}(x) = \frac{\kappa t^\beta}{(1 + |t^{-\gamma}x|^n)^m}, \quad d - nm = -1, \quad n, m, \kappa > 0 \right\},$$

and denote the diameter of  $\Omega$  defined as  $\sup \{|x - y| : x, y \in \Omega\}$  by  $\ell_\Omega$ . Then, for all  $t > 0$  satisfying  $t^\gamma < \ell_\Omega$ , we have

$$\|p_1^{(\alpha)}\|_{L^1(\mathbb{R}^d)} |\Omega| - t^{-(\beta+d\gamma)} \mathbb{H}_\Omega^{(\alpha)}(t) \leq t^\gamma \left( \lambda(\Omega) + \kappa w_{d-1} \text{Per}(\Omega) \gamma \ln \left( \frac{1}{t} \right) \right),$$

where

$$\lambda(\Omega) = |\Omega| \ell_\Omega^{-1} A_d \kappa + \kappa w_{d-1} \text{Per}(\Omega) \left( \ln(\ell_\Omega) + \int_0^1 \frac{dr r^d}{(1 + r^n)^m} \right).$$

In particular, we arrive at

$$\lim_{t \rightarrow 0+} \frac{\|p_1^{(\alpha)}\|_{L^1(\mathbb{R}^d)} |\Omega| - t^{-(\beta+d\gamma)} \mathbb{H}_\Omega^{(\alpha)}(t)}{t^\gamma \ln \left( \frac{1}{t} \right)} \leq \kappa \text{Per}(\Omega) w_{d-1} \gamma.$$

(iii) Let  $I_2 \subset I$  be such that for each  $\alpha \in I_2$ , there are functions  $\phi_\alpha : [0, 1] \rightarrow [0, \infty)$ ,  $J_\alpha : \mathbb{R}^d \rightarrow (0, \infty]$  and  $\Lambda_\alpha : \mathbb{R}^d \rightarrow (0, \infty]$  satisfying

$$(1.9) \quad \frac{p_t^{(\alpha)}(x)}{\phi_\alpha(t)} \leq \Lambda_\alpha(x), \quad x \neq 0, \quad 0 < t \leq 1,$$

$$(1.10) \quad \lim_{t \rightarrow 0+} \frac{p_t^{(\alpha)}(x)}{\phi_\alpha(t)} = J_\alpha(x), \quad x \neq 0,$$

and

$$(1.11) \quad \int_\Omega dx \int_{\Omega^c} dy J_\alpha(x - y) < \infty, \quad \int_\Omega dx \int_{\Omega^c} dy \Lambda_\alpha(x - y) < \infty.$$

Then,

$$\lim_{t \rightarrow 0+} \frac{t^{d\gamma+\beta} \|p_1^{(\alpha)}\|_{L^1(\mathbb{R}^d)} |\Omega| - \mathbb{H}_\Omega^{(\alpha)}(t)}{\phi_\alpha(t)} = \int_\Omega dx \int_{\Omega^c} dy J_\alpha(x - y).$$

We remark that the different small time behaviors provided in the foregoing theorem implicitly contains the fact that  $I_0, I_1$  and  $I_2$  are assumed to be disjoint subsets of  $I$  and at least one of them is not empty.

The key step to proving Theorem 1.1 consists on expressing the heat content  $\mathbb{H}_\Omega^{(\alpha)}(t)$  in terms of the set covariance function of  $\Omega$  defined in (2.3) below which as we shall see in the next section is linked to the perimeter of  $\Omega$ .

A classical example of heat kernels satisfying all the above assumptions are the transition probabilities corresponding to the rotational invariant  $\alpha$ -stable process whose main properties will be described in §4 below. There is an increasing interest in investigating small and large time behavior of functions related to the transition densities of a stable process and we refer the reader to [1, 2, 3, 4, 5, 6, 7] for recent developments concerning the heat trace, heat content and spectral heat content for bounded domains with smooth boundary and Schrödinger operators on  $\mathbb{R}^d$ .

The paper is organized as follows. In §2, we introduce the geometric objects associated with sets  $\Omega$  of finite perimeter. Namely, set covariance and bounded variation functions. In §3, we provide the proof of Theorem 1.1. In §4, an application of Theorem 1.1 is given when working with the heat kernels of a rotational invariant  $\alpha$ -stable process. Finally, in §5, the heat content over the unit ball related to the Poisson kernel (see (4.2) below) is investigated.

## 2. PRELIMINARIES: FUNCTIONS OF BOUNDED VARIATION, PERIMETER AND COVARIANCE FUNCTION.

In this section, we introduce a couple of geometric objects associated with the set  $\Omega$  under consideration which will play an important role in the proof of Theorem 1.1. The interested reader may consult [15], [19], [21] and [20] for further details on the matter and for the proofs of the many results to be given in this section.

**Definition 2.1.** Let  $G \subseteq \mathbb{R}^d$  be an open set and  $f : G \rightarrow \mathbb{R}$ ,  $f \in L^1(G)$ . The total variation of  $f$  in  $G$  is defined by

$$V(f, G) = \sup \left\{ \int_G dx f(x) \mathbf{div} \varphi(x) : \varphi \in C_c^1(G, \mathbb{R}^d), \|\varphi\|_\infty \leq 1 \right\}.$$

We set  $BV(G) = \{f \in L^1(G) : V(f, G) < \infty\}$  to denote the set of functions of bounded variation.

The directional derivative of  $f$  in  $G$  in the direction  $u \in S^{d-1}$  is

$$V_u(f, G) = \sup \left\{ \int_G dx f(x) \langle \nabla \varphi(x), u \rangle : \varphi \in C_c^1(G, \mathbb{R}^d), \|\varphi\|_\infty \leq 1 \right\}.$$

If  $\Omega \in \mathcal{L}(\mathbb{R}^d)$ , we call  $V(\mathbb{1}_\Omega, \mathbb{R}^d)$  the perimeter of  $\Omega$  and we denote this quantity by  $Per(\Omega)$ . In addition,  $V_u(\Omega)$  will denote for simplicity the quantity  $V_u(\mathbb{1}_\Omega, \mathbb{R}^d)$ .

In order to gain an insight into functions of bounded variation, we proceed to provide some classical examples.

**Example 2.1.** Let  $G \subseteq \mathbb{R}^d$  be an open set and consider

$$W^{1,1}(G) = \left\{ f \in L^1(G) : \frac{\partial f}{\partial x_j} \in L^1(G) \right\}.$$

Then  $W^{1,1}(G) \subseteq BV(G)$ . To prove this, it suffices to see that by integration by parts formula, we have for any  $\varphi \in C_c^1(G, \mathbb{R}^d)$  with  $\|\varphi\|_\infty \leq 1$  that

$$\int_G dx f(x) \mathbf{div} \varphi(x) = - \int_G dx \langle \nabla f(x), \varphi(x) \rangle \leq \int_G dx |\nabla f(x)| < \infty.$$

Thus,

$$(2.1) \quad V(f, G) \leq \int_G dx |\nabla f(x)|.$$

It is worth mentioning that equality indeed happens in (2.1) and we refer the reader to [21] for the proof.

The following example tells us that the perimeter of a bounded set and the  $(d-1)$ -Hausdorff measure of the boundary are related provided that the boundary is smooth enough.

**Example 2.2.** Let  $\Omega \in \mathcal{L}(\mathbb{R}^d)$  be a bounded open set with  $C^2$  boundary  $\partial\Omega$  and  $\mathcal{H}^{d-1}(\partial\Omega) < \infty$ . Then, we claim that  $\mathbb{1}_\Omega \in BV(\Omega)$  and

$$Per(\Omega) = V(\mathbb{1}_\Omega, \mathbb{R}^d) = \mathcal{H}^{d-1}(\partial\Omega).$$

To see this, we apply the divergence theorem to any  $\varphi \in C_c^1(\mathbb{R}^d, \mathbb{R}^d)$  with  $\|\varphi\|_\infty \leq 1$  to obtain

$$(2.2) \quad \int_\Omega dx \mathbf{div} \varphi(x) = \int_{\partial\Omega} \langle \varphi(x), n(x) \rangle \mathcal{H}^{d-1}(dx),$$

where  $n(x)$  is the unit normal along  $\partial\Omega$ . The last identity implies  $Per(\Omega) \leq \mathcal{H}^{d-1}(\partial\Omega)$  since  $\langle \varphi(x), n(x) \rangle \leq |\varphi(x)| |n(x)| \leq 1$ .

On the other hand, the facts that  $\Omega$  is bounded and  $\partial\Omega$  is  $C^2$  allow us to construct a compact set  $K$  such that  $\Omega \subset K$  and a vector field  $\varphi_\Omega \in C_c^1(\mathbb{R}^d, \mathbb{R}^d)$  with  $\|\varphi_\Omega\|_\infty \leq 1$  satisfying  $\varphi_\Omega(x) = n(x)$  for  $x \in \partial\Omega$  and  $\varphi_\Omega(x) = 0$  for  $x \in K^c$ . Hence, an application of (2.2) yields

$$\int_{\Omega} dx \operatorname{div} \varphi_\Omega(x) = \int_{\partial\Omega} \langle n(x), n(x) \rangle \mathcal{H}^{d-1}(dx) = \mathcal{H}^{d-1}(\partial\Omega),$$

which in turn finishes the proof of our claim.

The following geometric object will be essential in order to investigate the small time behavior of the heat content  $\mathbb{H}_\Omega^{(\alpha)}(t)$ .

**Definition 2.2 (Set covariance function).** Let  $\Omega \in \mathcal{L}(\mathbb{R}^d)$  have finite Lebesgue measure. The covariance function of  $\Omega$  is denoted by  $g_\Omega$  and defined for each  $y \in \mathbb{R}^d$  by

$$(2.3) \quad g_\Omega(y) = |\Omega \cap (\Omega + y)| = \int_{\mathbb{R}^d} dx \mathbb{1}_\Omega(x) \mathbb{1}_\Omega(x - y).$$

We now proceed to mention some analytic properties associated with the set covariance function  $g_\Omega$ .

**Proposition 2.1.** Let  $\Omega \subset \mathbb{R}^d$  be a Lebesgue measurable set with  $|\Omega| < \infty$  and  $g_\Omega$  its corresponding covariance function. Then,

- a) For all  $y \in \mathbb{R}^d$ ,  $0 \leq g_\Omega(y) \leq g_\Omega(0) = |\Omega|$ .
- b) For all  $y \in \mathbb{R}^d$ ,  $g_\Omega(y) = g_\Omega(-y)$ .
- c)  $\int_{\mathbb{R}^d} dy g_\Omega(y) = |\Omega|^2$ .
- d)  $g_\Omega$  is compactly supported. In fact, for all  $y \in \mathbb{R}^d$  with  $|y| \geq \ell_\Omega = \sup \{|x - y| : x, y \in \Omega\}$ , we have  $g_\Omega(y) = 0$ .
- e)  $g_\Omega$  is uniformly continuous over  $\mathbb{R}^d$  and  $\lim_{|y| \rightarrow \infty} g_\Omega(y) = 0$ .

The following propositions reveal the link among functions of bounded variation, directional variation and sets of finite perimeter. The proof of these results can be found in [18].

**Proposition 2.2.** Let  $G$  be an open subset of  $\mathbb{R}^d$  and consider  $f \in L^1(\mathbb{R}^d)$ . Then,  $V(f, G)$  is finite if and only if the directional variation  $V_u(f, G)$  is finite for every direction  $u \in S^{d-1}$  and

$$(2.4) \quad V(f, G) = \frac{1}{2w_{d-1}} \int_{S^{d-1}} \mathcal{H}^{d-1}(du) V_u(f, G).$$

In particular, for any  $\Omega \in \mathcal{L}(\mathbb{R}^d)$  with finite perimeter, we have

$$(2.5) \quad \operatorname{Per}(\Omega) = \frac{1}{2w_{d-1}} \int_{S^{d-1}} \mathcal{H}^{d-1}(du) V_u(\Omega).$$

**Proposition 2.3.** Let  $\Omega \in \mathcal{L}(\mathbb{R}^d)$  be such that  $|\Omega|$  is finite and consider  $g_\Omega$  its corresponding covariance function and  $u \in S^{d-1}$ . The following assertions are equivalent.

- (i)  $V_u(\Omega)$  is finite.
- (ii)  $\lim_{r \rightarrow 0} \frac{g_\Omega(0) - g_\Omega(ru)}{|r|}$  exists and is finite.
- (iii) The real valued function  $g_\Omega^u(r) = g_\Omega(ru)$  is Lipschitz. Moreover,

$$\operatorname{Lip}(g_\Omega^u) = \lim_{r \rightarrow 0} \frac{g_\Omega(0) - g_\Omega(ru)}{|r|} = \frac{V_u(\Omega)}{2}.$$

**Proposition 2.4.** Let  $\Omega \in \mathcal{L}(\mathbb{R}^d)$  be a bounded set with  $\operatorname{Per}(\Omega) < \infty$  and consider  $g_\Omega$  its corresponding covariance function. Then,

i)  $g_\Omega$  is Lipschitz with

$$\text{Lip}(g_\Omega) = \frac{1}{2} \sup_{u \in S^{d-1}} V_u(\Omega) \leq \frac{1}{2} \text{Per}(\Omega).$$

ii) For each  $u \in S^{d-1}$  and  $r > 0$ ,

$$(g_\Omega^u)'(0+) = \lim_{r \rightarrow 0+} \frac{g_\Omega(ru) - g_\Omega(0)}{r}$$

exists and is finite. Moreover,

$$\text{Per}(\Omega) = -\frac{1}{w_{d-1}} \int_{S^{d-1}} \mathcal{H}^{d-1}(du) (g_\Omega^u)'(0+).$$

### 3. PROOF OF THEOREM 1.1

We begin this section by rewriting  $\mathbb{H}_\Omega^{(\alpha)}(t)$  in terms of the set covariance function  $g_\Omega$ . By appealing to Fubini's Theorem and performing a simple change of variable, we have based on (1.4) and (2.3) that

$$\mathbb{H}_\Omega^{(\alpha)}(t) = \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy p_t^{(\alpha)}(x-y) \mathbb{1}_\Omega(y) \mathbb{1}_\Omega(x) = \int_{\mathbb{R}^d} dz p_t^{(\alpha)}(z) g_\Omega(z).$$

By using the scaling property (1.1) and the change of variable  $w = t^{-\gamma}z$ , we arrive at

$$\begin{aligned} (3.1) \quad t^{-(d\gamma+\beta)} \mathbb{H}_\Omega^{(\alpha)}(t) &= \int_{\mathbb{R}^d} dw p_1^{(\alpha)}(w) g_\Omega(t^\gamma w) \\ &= g_\Omega(0) \|p_1^{(\alpha)}\|_{L^1(\mathbb{R}^d)} + \int_{\mathbb{R}^d} dw p_1^{(\alpha)}(w) (g_\Omega(t^\gamma w) - g_\Omega(0)). \end{aligned}$$

Hence, we have shown that

$$(3.2) \quad \int_{\mathbb{R}^d} dw p_1^{(\alpha)}(w) (g_\Omega(0) - g_\Omega(t^\gamma w)) = g_\Omega(0) \|p_1^{(\alpha)}\|_{L^1(\mathbb{R}^d)} - t^{-(d\gamma+\beta)} \mathbb{H}_\Omega^{(\alpha)}(t),$$

where  $g_\Omega(0) = |\Omega|$  by Proposition 2.1. Next, with the aid of (3.2), we start the proof of Theorem 1.1.

**Proof of part (i) of Theorem 1.1:** Let us define

$$(3.3) \quad F_\Omega^{(\alpha)}(t) = \int_{\mathbb{R}^d} dw p_1^{(\alpha)}(w) (g_\Omega(0) - g_\Omega(t^\gamma w)).$$

Now, polar coordinates and the fact that  $p_t^{(\alpha)}(x) = p_t^{(\alpha)}(|x| e_d)$  allow us to express  $F_\Omega^{(\alpha)}(t)$  as follows.

$$\begin{aligned} (3.4) \quad F_\Omega^{(\alpha)}(t) &= \int_0^\infty dr r^{d-1} p_1^{(\alpha)}(r e_d) \int_{S^{d-1}} \mathcal{H}^{d-1}(du) (g_\Omega(0) - g_\Omega(t^\gamma r u)) \\ &= t^\gamma \int_0^\infty dr r^d p_1^{(\alpha)}(r e_d) M_\Omega(t, r), \end{aligned}$$

where

$$(3.5) \quad M_\Omega(t, r) = \int_{S^{d-1}} \mathcal{H}^{d-1}(du) \left( \frac{g_\Omega(0) - g_\Omega(t^\gamma r u)}{t^\gamma r} \right).$$

We first proceed to prove (1.8). Notice that by Proposition 2.4, we have

$$(3.6) \quad \left| \frac{g_\Omega(t^\gamma r u) - g_\Omega(0)}{t^\gamma r} \right| \leq \frac{1}{2} \text{Per}(\Omega) \in L^1(S^{d-1}).$$

Therefore, it follows from the last inequality, part (ii) of Proposition 2.4 and the Lebesgue Dominated convergence Theorem that

$$(3.7) \quad \lim_{t \rightarrow 0+} M_\Omega(t, r) = w_{d-1} \text{Per}(\Omega).$$

On the other hand, it follows from (3.5) and (3.6) that

$$(3.8) \quad r^d p_1^{(\alpha)}(r e_d) M_\Omega(t, r) \leq \frac{1}{2} A_d \text{Per}(\Omega) r^d p_1^{(\alpha)}(r e_d).$$

Notice that  $r^d p_1^{(\alpha)}(r e_d) \in L^1((0, \infty))$  because the assumption  $|\cdot| p_1^{(\alpha)}(\cdot) \in L^1(\mathbb{R}^d)$  and polar coordinates imply that

$$(3.9) \quad A_d \int_0^\infty dr r^d p_1^{(\alpha)}(r e_d) = \int_{\mathbb{R}^d} dw |w| p_1^{(\alpha)}(w) < \infty.$$

Hence, by combining the Lebesgue Dominated convergence Theorem together with the limit (3.7) and inequality (3.8), we conclude by appealing to the identities (3.2) and (3.3) that

$$\begin{aligned} \lim_{t \rightarrow 0+} \frac{g_\Omega(0) \|p_1^{(\alpha)}\|_{L^1(\mathbb{R}^d)} - t^{-(d\gamma+\beta)} \mathbb{H}_\Omega^{(\alpha)}(t)}{t^\gamma} &= \lim_{t \rightarrow 0+} \frac{F_\Omega^{(\alpha)}(t)}{t^\gamma} \\ &= w_{d-1} \text{Per}(\Omega) \int_0^\infty dr r^d p_1^{(\alpha)}(r e_d). \end{aligned}$$

Regarding the inequality (1.7), by appealing to the definition of  $M_\Omega(t, r)$  provided in (3.5) and part (iii) of Proposition 2.3, we derive from identity (3.4) that

$$F_\Omega^{(\alpha)}(t) \leq t^\gamma \int_0^\infty dr r^d p_1^{(\alpha)}(r e_d) \left( \frac{1}{2} \int_{S^{d-1}} \mathcal{H}^{d-1}(du) V_u(\Omega) \right).$$

Thus, it follows from (2.5) that

$$F_\Omega^{(\alpha)}(t) \leq t^\gamma w_{d-1} \text{Per}(\Omega) \int_0^\infty dr r^d p_1^{(\alpha)}(r e_d).$$

Finally, observe that (3.2) and (3.3) lead to the desired inequality (1.7) and this finishes the proof of part (i) of Theorem 1.1.  $\square$

The proof of part (ii) cannot follow the same outline of part (i) because for every  $d, n, m$  satisfying  $d - nm = -1$  and  $n, m > 0$ , we have

$$\int_1^\infty dr \frac{r^d}{(1+r^n)^m} \geq 2^{-m} \int_1^\infty dr \frac{r^d}{r^{nm}} = 2^{-m} \int_1^\infty dr r^{-1} = \infty,$$

where we have used that  $1 + r^n \leq 2r^n$  for all  $r \geq 1$ . Thus, the divergence of the last integral in turn implies by polar coordinates that  $|\cdot| p_t^{(\alpha)}(\cdot) \notin L^1(\mathbb{R}^d)$  (see (3.9)) for every heat kernel  $p_t^{(\alpha)}(\cdot)$  satisfying assumptions in part (ii) of Theorem 1.1.

**Proof of part (ii) of Theorem 1.1:** We recall that along the proof, we assume that the heat kernels  $p_t^{(\alpha)}(x)$  are explicitly given by

$$(3.10) \quad p_t^{(\alpha)}(x) = \frac{\kappa t^\beta}{(1 + |t^{-\gamma} x|^n)^m},$$

with  $d - nm = -1$  and  $n, m, \kappa > 0$ . By part d) in Proposition 2.1, we know that  $g_\Omega(y) = 0$  if  $|y| > \ell_\Omega$  with  $\ell_\Omega$  being the diameter of  $\Omega$ . Therefore, we arrive at the following decomposition of

the heat content.

$$(3.11) \quad \begin{aligned} t^{-(d\gamma+\beta)} \mathbb{H}_\Omega^{(\alpha)}(t) &= \int_{\mathbb{R}^d} dw p_1^{(\alpha)}(w) g_\Omega(t^\gamma w) \\ &= g_\Omega(0) \|p_1^{(\alpha)}\|_{L^1(\mathbb{R}^d)} - F_1(t) - t^\gamma F_2(t), \end{aligned}$$

where

$$F_1(t) = g_\Omega(0) \int_{|w| \geq \ell_\Omega t^{-\gamma}} dw p_1^{(\alpha)}(w)$$

and

$$\begin{aligned} F_2(t) &= \int_{|w| < \ell_\Omega t^{-\gamma}} dw p_1^{(\alpha)}(w) \left( \frac{g_\Omega(0) - g_\Omega(t^\gamma w)}{t^\gamma} \right) \\ &= \int_{S^{d-1}} \mathcal{H}^{d-1}(du) \int_0^{\ell_\Omega t^{-\gamma}} dr r^d p_1^{(\alpha)}(r e_d) \left( \frac{g_\Omega(0) - g_\Omega(t^\gamma r u)}{t^\gamma r} \right). \end{aligned}$$

Now, due to the fact that  $1 + r^n > r^n$  and  $d - nm = -1$ , we obtain by appealing to polar coordinates and the explicit form of the heat kernels (3.10) that

$$(3.12) \quad F_1(t) = g_\Omega(0) A_d \kappa \int_{\ell_\Omega t^{-\gamma}}^\infty \frac{dr r^{d-1}}{(1 + r^n)^m} \leq A_d \kappa \int_{\ell_\Omega t^{-\gamma}}^\infty dr r^{-2} = |\Omega| A_d \kappa \ell_\Omega^{-1} t^\gamma.$$

On the other hand, by Proposition 2.3, we obtain that

$$0 \leq \frac{g_\Omega(0) - g_\Omega(t^\gamma r u)}{t^\gamma r} \leq \frac{1}{2} V_u(\Omega).$$

Thus, we deduce by (2.5) and (3.10) that

$$(3.13) \quad F_2(t) \leq \text{Per}(\Omega) w_{d-1} \kappa \int_0^{\ell_\Omega t^{-\gamma}} \frac{dr r^d}{(1 + r^n)^m}.$$

Now, we observe that

$$\int_0^{\ell_\Omega t^{-\gamma}} \frac{dr r^d}{(1 + r^n)^m} = \int_0^1 \frac{dr r^d}{(1 + r^n)^m} + \int_1^{\ell_\Omega t^{-\gamma}} \frac{dr r^d}{(1 + r^n)^m},$$

as long as  $t^\gamma < \ell_\Omega$ . Therefore, we arrive by using once more that  $d - nm = -1$  and  $(1 + r^n)^m \geq r^{nm}$  at

$$\int_1^{\ell_\Omega t^{-\gamma}} \frac{dr r^d}{(1 + r^n)^m} \leq \int_1^{\ell_\Omega t^{-\gamma}} dr r^{-1} = \ln(\ell_\Omega) + \gamma \ln\left(\frac{1}{t}\right).$$

Hence, by (3.13), we have proved that

$$(3.14) \quad F_2(t) \leq \kappa \text{Per}(\Omega) w_{d-1} \left( \int_0^1 \frac{dr r^d}{(1 + r^n)^m} + \ln(\ell_\Omega) + \gamma \ln\left(\frac{1}{t}\right) \right).$$

Therefore, by combining the previous inequalities (3.12) and (3.14) together with the identity

$$g_\Omega(0) \|p_1^{(\alpha)}\|_{L^1(\mathbb{R}^d)} - t^{-(\beta+d\gamma)} \mathbb{H}_\Omega^{(\alpha)}(t) = F_1(t) + t^\gamma F_2(t),$$

we arrive at the desired result and this finishes the proof of part (ii) of Theorem 1.1.  $\square$

**Proof of part (iii) of Theorem 1.1:** By using that  $\mathbb{1}_\Omega(\cdot) = 1 - \mathbb{1}_{\Omega^c}(\cdot)$ , we have

$$(3.15) \quad g_\Omega(y) = \int_{\mathbb{R}^d} dx \mathbb{1}_\Omega(x) \mathbb{1}_\Omega(x + y) = |\Omega| - h_\Omega(y),$$

where

$$h_\Omega(y) = \int_{\mathbb{R}^d} dx \mathbb{1}_\Omega(x) \mathbb{1}_{\Omega^c}(x + y).$$



This implies

$$\mathbb{H}_\Omega^{(\alpha)}(t) = \int_{\mathbb{R}^d} dz p_t^{(\alpha)}(z) g_\Omega(z) = |\Omega| \|p_1^{(\alpha)}\|_{L^1(\mathbb{R}^d)} t^{\beta+d\gamma} - \int_{\mathbb{R}^d} dz p_t^{(\alpha)}(z) h_\Omega(z).$$

It is easy to show by Fubini's Theorem that

$$\int_{\mathbb{R}^d} dz p_t^{(\alpha)}(z) h_\Omega(z) = \int_\Omega dx \int_{\Omega^c} dy p_t^{(\alpha)}(x-y).$$

Thus, we arrive at

$$(3.16) \quad \lim_{t \rightarrow 0+} \frac{|\Omega| \|p_1^{(\alpha)}\|_{L^1(\mathbb{R}^d)} t^{\beta+d\gamma} - \mathbb{H}_\Omega^{(\alpha)}(t)}{\phi_\alpha(t)} = \lim_{t \rightarrow 0+} \int_\Omega dx \int_{\Omega^c} dy \frac{p_t^{(\alpha)}(x-y)}{\phi_\alpha(t)}.$$

The conditions (1.9), (1.10) and (1.11) are given to apply the Lebesgue dominated convergence Theorem to the right hand side of (3.16) which in turn completes the proof of Theorem 1.1.

#### 4. APPLICATIONS TO ROTATIONAL INVARIANT $\alpha$ -STABLE PROCESSES, $0 < \alpha < 2$ .

The heat kernels of rotational invariant  $\alpha$ -stable processes are probability densities

$$\left\{ p_t^{(\alpha)}(\cdot) : \mathbb{R}^d \rightarrow [0, \infty] : t \geq 0, \alpha \in I = (0, 2) \right\},$$

which are completely determined by their Fourier transform and they satisfy the following properties.

- (i) For all  $\alpha \in (0, 2)$ ,  $t > 0$  and  $x \in \mathbb{R}^d$ , we have that  $\widehat{p_t^{(\alpha)}}(x) = e^{-t|x|^\alpha}$ . As a result of this Fourier transform, we see that  $p_t^{(\alpha)}(x)$  is radial and  $\|p_t^{(\alpha)}\|_{L^1(\mathbb{R}^d)} = 1$ .
- (ii) Scaling Property: for each integer  $d \geq 2$  and  $\alpha \in (0, 2)$ ,

$$(4.1) \quad p_t^{(\alpha)}(x) = t^{-\frac{d}{\alpha}} p_1^{(\alpha)}(t^{-\frac{1}{\alpha}} x).$$

That is, they satisfy the scaling property (1.1) with  $\beta = -\frac{d}{\alpha}$  and  $\gamma = \frac{1}{\alpha}$ , where for these values we obtain  $\beta + d\gamma = 0$ .

The transition densities  $p_t^{(\alpha)}(x-y)$  are known to have an explicit expression only for  $\alpha = 1$ . In fact, for  $\alpha = 1$ , the function  $p_t^{(1)}(x-y)$  is called the Poisson heat kernel and it is given by

$$(4.2) \quad p_t^{(1)}(x-y) = \frac{k_d t}{\left(t^2 + |x-y|^2\right)^{(d+1)/2}},$$

where

$$(4.3) \quad k_d = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}}.$$

However, for the purposes of this paper, we only need to make use of the following two facts about  $p_t^{(\alpha)}(x-y)$  for all  $\alpha \in (0, 2)$ . First, there exists  $c_{\alpha,d} > 0$  such that

$$(4.4) \quad c_{\alpha,d}^{-1} \min \left\{ t^{-d/\alpha}, \frac{t}{|x-y|^{d+\alpha}} \right\} \leq p_t^{(\alpha)}(x-y) \leq c_{\alpha,d} \min \left\{ t^{-d/\alpha}, \frac{t}{|x-y|^{d+\alpha}} \right\},$$

for all  $x, y \in \mathbb{R}^d$  and  $t > 0$  (see [13]). Secondly, according to [12, Theorem 2.1], we have

$$(4.5) \quad \lim_{t \rightarrow 0+} \frac{p_t^{(\alpha)}(x-y)}{t} = \frac{C_{\alpha,d}}{|x-y|^{d+\alpha}},$$

for all  $x \neq y$ , where

$$(4.6) \quad C_{\alpha,d} = \alpha 2^{\alpha-1} \pi^{-1-\frac{d}{2}} \sin\left(\frac{\pi\alpha}{2}\right) \Gamma\left(\frac{d+\alpha}{2}\right) \Gamma\left(\frac{\alpha}{2}\right).$$

One interesting aspect about the heat kernel  $p_t^{(\alpha)}(x-y)$  is that it can be written in terms of the Gaussian kernel (1.5). Namely, for each  $\alpha \in (0, 2)$ , there exist probability functions denoted by  $\{\eta_t^{(\alpha/2)}\}_{t>0}$  satisfying

$$(4.7) \quad p_t^{(\alpha)}(x-y) = \int_0^\infty ds p_s^{(2)}(x-y) \eta_t^{(\alpha/2)}(s).$$

Furthermore, in [2] is proved by means of probabilistic techniques that  $s^\lambda \eta_1^{(\alpha/2)}(s) \in L^1((0, \infty))$  if and only if  $-\infty < \lambda < \frac{\alpha}{2}$  and

$$(4.8) \quad \int_0^\infty ds s^\lambda \eta_1^{(\alpha/2)}(s) = \frac{\Gamma(1 - \frac{2\lambda}{\alpha})}{\Gamma(1 - \lambda)}.$$

The aforementioned estimates gives the following result.

**Lemma 4.1.**  $|\cdot| p_1^{(\alpha)}(\cdot) \in L^1(\mathbb{R}^d)$  if and only if  $r^d p_1^{(\alpha)}(re_d) \in L^1((0, \infty))$  if and only if  $\alpha \in (1, 2)$ . Moreover,

$$(4.9) \quad \int_0^\infty dr r^d p_1^{(\alpha)}(r) = \Gamma\left(\frac{d+1}{2}\right) \pi^{-\frac{d+1}{2}} \Gamma\left(1 - \frac{1}{\alpha}\right).$$

*Proof.* Observe that (4.7) yields

$$(4.10) \quad \int_0^\infty dr r^d p_1^{(\alpha)}(re_d) = (4\pi)^{-d/2} \int_0^\infty ds \eta_1^{(\alpha/2)}(s) s^{-d/2} \int_0^\infty dr r^d \exp\left(-\frac{r^2}{4s}\right).$$

Next, the change of variables  $w = \frac{r^2}{4s}$  shows that

$$\int_0^\infty dr r^d \exp\left(-\frac{r^2}{4s}\right) = 4^{d/2} s^{\frac{d+1}{2}} \Gamma\left(\frac{d+1}{2}\right).$$

Thus, the desired result is obtained by replacing the last identity into the integral (4.10) and using (4.8) with  $\lambda = 1/2$ .  $\square$

The following theorem is the main result concerning the small time behavior of the heat content of  $\Omega$  in  $\mathbb{R}^d$  when dealing with the heat kernels of rotationally invariant  $\alpha$ -stable processes. It is remarkable that part a) and c) of the next result are stronger than Theorem 1.2 in [1] since uniformly  $C^{1,1}$ -regular bounded domains have according to Example 2.2 finite perimeter.

**Theorem 4.1.** Consider  $\Omega \in \mathcal{L}(\mathbb{R}^d)$  a bounded set with  $Per(\Omega) < \infty$  and  $d \geq 2$  integer.

(a) Let  $\alpha \in (1, 2)$ . Then, we have for all  $t > 0$  that

$$|\Omega| - \mathbb{H}_\Omega^{(\alpha)}(t) \leq \frac{t^{\frac{1}{\alpha}}}{\pi} \Gamma\left(1 - \frac{1}{\alpha}\right) Per(\Omega).$$

Furthermore,

$$\lim_{t \rightarrow 0+} \frac{|\Omega| - \mathbb{H}_\Omega^{(\alpha)}(t)}{t^{\frac{1}{\alpha}}} = \frac{1}{\pi} \Gamma\left(1 - \frac{1}{\alpha}\right) Per(\Omega).$$

(b) For  $\alpha = 1$ , we have for all  $t < \ell_\Omega = \sup\{|x-y| : x, y \in \Omega\}$  that

$$|\Omega| - \mathbb{H}_\Omega^{(1)}(t) \leq t \left( \lambda(\Omega) + \frac{1}{\pi} Per(\Omega) \ln\left(\frac{1}{t}\right) \right).$$

Here,

$$\lambda(\Omega) = |\Omega| \ell_\Omega^{-1} A_d \kappa_d + \frac{Per(\Omega)}{\pi} \left( \ln(\ell_\Omega) + \int_0^1 \frac{dr r^d}{(1+r^2)^{\frac{d+1}{2}}} \right),$$

and  $\kappa_d$  and  $A_d$  as given in (4.3) and (1.3), respectively. In particular, we arrive at

$$(4.11) \quad \lim_{t \rightarrow 0+} \frac{|\Omega| - \mathbb{H}_\Omega^{(1)}(t)}{t \ln\left(\frac{1}{t}\right)} \leq \frac{1}{\pi} \text{Per}(\Omega).$$

In addition, the inequality (4.11) is sharp. That is, if  $B$  is the unit ball in  $\mathbb{R}^d$ , then

$$(4.12) \quad \lim_{t \rightarrow 0+} \frac{|B| - \mathbb{H}_B^{(1)}(t)}{t \ln\left(\frac{1}{t}\right)} = \frac{1}{\pi} \text{Per}(B).$$

(c) For  $0 < \alpha < 1$ ,

$$\lim_{t \rightarrow 0+} \frac{|\Omega| - \mathbb{H}_\Omega^{(\alpha)}(t)}{t} = C_{\alpha,d} \mathcal{P}_\alpha(\Omega),$$

where

$$(4.13) \quad \mathcal{P}_\alpha(\Omega) = \int_\Omega \int_{\Omega^c} \frac{dx dy}{|x - y|^{d+\alpha}},$$

with  $C_{\alpha,d}$  as given in (4.6).

**Remark 4.1.** We point out that  $\mathcal{P}_\alpha(\Omega)$  defined in (4.13) is called the  $\alpha$ -perimeter and it turns out to be linked with celebrated Hardy and isoperimetric inequalities. We refer the reader to the papers of Z. Q. Chen, R. Song [14] and R. L. Frank, R. Seiringer [16] for further results involving this quantity. In fact, it is shown in [16] that there exists  $\lambda_{d,\alpha} > 0$  such that

$$|\Omega|^{(d-\alpha)/d} \leq \lambda_{d,\alpha} \mathcal{P}_\alpha(\Omega),$$

with equality if and only if  $\Omega$  is a ball. It is also proved in [19] and [17] that

$$\begin{aligned} \lim_{\alpha \downarrow 0} \alpha \mathcal{P}_\alpha(\Omega) &= d |B_1(0)| |\Omega|, \\ \lim_{\alpha \uparrow 1} (1 - \alpha) \mathcal{P}_\alpha(\Omega) &= K_d \mathcal{H}^{d-1}(\partial\Omega), \end{aligned}$$

for some  $K_d > 0$ .

**Proof of Theorem 4.1:** This theorem is basically a consequence of applying Theorem 1.1 with  $\beta = -\frac{d}{\alpha}$  and  $\gamma = \frac{1}{\alpha}$ . For the proof of part a), we apply part (i) of Theorem 1.1 with  $I_1 = (1, 2)$  due to Lemma 4.1. Regarding part (c), we apply part (iii) of Theorem 1.1 with (based on (4.4) and (4.5))  $\phi_\alpha(t) = t$ ,  $\Lambda_\alpha(x) = c_{\alpha,d} |x|^{-(d+\alpha)}$  and  $J_\alpha(x) = C_{\alpha,d} |x|^{-(d+\alpha)}$ , where we need to make use of the fact that  $\mathcal{P}_\alpha(\Omega) < \infty$  if and only if  $\alpha \in (0, 1)$  provided that  $\text{Per}(\Omega) < \infty$ , according to Corollary 2.13 in [19].

As far as part (b), we appeal to part (ii) of Theorem 1.1 with  $\kappa = \kappa_d$ ,  $n = 2$  and  $m = \frac{d+1}{2}$ . Notice that according to (1.3) and (4.3), we have

$$(4.14) \quad \kappa_d w_{d-1} = \Gamma\left(\frac{d+1}{2}\right) \pi^{-\frac{d+1}{2}} \cdot \frac{\pi^{\frac{d-1}{2}}}{\Gamma\left(1 + \frac{d-1}{2}\right)} = \frac{1}{\pi}.$$

Concerning the limit (4.12), we could apply Theorem 1.2 in [1], however to prove that theorem, local coordinates around each point of the boundary are required which makes the proof very complicated. For this reason, in order to make this presentation as clear as possible, we provide a somewhat simple proof for (4.12) and we devote the next section to it.  $\square$

### 5. HEAT CONTENT BEHAVIOR FOR THE POISSON KERNEL (4.2) OVER THE UNIT BALL.

We want to study the small time asymptotic behavior of

$$\mathbb{H}_B^{(1)}(t) = \int_{\mathbb{R}^d} dx p_t^{(1)}(x) g_B(x) = \int_{\mathbb{R}^d} dx p_1^{(1)}(x) g_B(tx),$$

where  $B \subset \mathbb{R}^d$  is the unit ball and  $p_t^{(1)}(x)$  is the heat kernel described in (4.2). Observe that  $B + z = B_1(z) = \{x \in \mathbb{R}^d : |x - z| < 1\}$  so that  $g_B(z) = |B \cap (B + z)|$  represents the volume of the intersection of two balls of radii one. It is also geometrically clear that  $g_B(z) = g_B(Tz)$  for any orthonormal linear transformation  $T$  on  $\mathbb{R}^d$ , which implies that  $g_B$  is radial so that

$$(5.1) \quad g_B(z) = g_B(|z| e_d),$$

with  $e_d = (0, \dots, 0, 1) \in \mathbb{R}^d$ .

The following lemma provides a formula for the volume of the intersection of two unit balls in  $\mathbb{R}^d$ .

**Lemma 5.1.** *Let  $d \geq 2$  be an integer and  $0 \leq a \leq 2$ . Let  $B = B_1(0)$  and  $B(a) = B_1(a e_d)$ . Then, we have*

$$(5.2) \quad g_B(a e_d) = |B \cap B(a)| = 2 A_{d-1} \Theta \left( \sqrt{1 - \frac{a^2}{4}} \right) - a w_{d-1} \left( 1 - \frac{a^2}{4} \right)^{\frac{d-1}{2}},$$

where

$$\Theta(z) = \int_0^{\arcsin(z)} d\theta \sin^{d-2}(\theta) \cos^2(\theta),$$

for  $0 \leq z \leq 1$  and  $w_{d-1}, A_{d-1}$  as defined in (1.3). In particular, by taking  $a = 0$ , we obtain

$$(5.3) \quad \Theta(1) = \frac{|B|}{2A_{d-1}}.$$

*Proof.* If  $a = 0$ , the result is obvious. Assume  $0 < a \leq 2$ . We start by representing every point  $x \in \mathbb{R}^d$  as  $x = (\bar{x}, x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ . It is not difficult to see that under these coordinates, we have

$$B \cap B(a) = \left\{ (\bar{x}, x_d) \in \mathbb{R}^d : |\bar{x}| \leq \sqrt{1 - \frac{a^2}{4}}, a - \sqrt{1 - |\bar{x}|^2} < x_d < \sqrt{1 - |\bar{x}|^2} \right\}.$$

Therefore, by setting  $\ell = \sqrt{1 - \frac{a^2}{4}}$ , we arrive at

$$\begin{aligned} |B \cap B(a)| &= \int_{B_\ell(0) \subset \mathbb{R}^{d-1}} d\bar{x} \int_{a - \sqrt{1 - |\bar{x}|^2}}^{\sqrt{1 - |\bar{x}|^2}} dx_d = 2 \int_{B_\ell(0) \subset \mathbb{R}^{d-1}} d\bar{x} \sqrt{1 - |\bar{x}|^2} - a |B_\ell(0)| \\ &= 2A_{d-1} \int_0^\ell dr r^{d-2} \sqrt{1 - r^2} - a w_{d-1} \ell^{d-1}, \end{aligned}$$

where we have appealed to polar coordinates to obtain the last equality. Thus, the desired identity (5.2) follows from the last expression and by performing the change of variable  $r = \sin(\theta)$  in the integral term.  $\square$

Since  $g_B(y) = 0$  when  $|y| \geq 2$  by Proposition 2.1 and  $g_B(y)$  is radial (see (5.1)), we obtain by combining the identities  $A_d = \text{Per}(B)$  and (4.14) together with Lemma 5.1 the following

decomposition for  $\mathbb{H}_B^{(1)}(t)$ .

$$(5.4) \quad \begin{aligned} \mathbb{H}_B^{(1)}(t) &= A_d \int_0^{2t^{-1}} dr r^{d-1} p_1^{(1)}(re_d) g_B(tr e_d) \\ &= N_1(t) - A_d \kappa_d w_{d-1} t N_2(t) = N_1(t) - \frac{1}{\pi} \text{Per}(B) t N_2(t), \end{aligned}$$

where

$$N_1(t) = 2A_d A_{d-1} \kappa_d \int_0^{2t^{-1}} \frac{dr r^{d-1}}{(1+r^2)^{\frac{d+1}{2}}} \Theta \left( \sqrt{1 - \frac{t^2 r^2}{4}} \right)$$

and

$$(5.5) \quad N_2(t) = \int_0^{2t^{-1}} \frac{dr r^d}{(1+r^2)^{\frac{d+1}{2}}} \left( 1 - \frac{t^2 r^2}{4} \right)^{\frac{d-1}{2}}.$$

Because  $\Theta$  given in Lemma 5.1 is an increasing function and  $2A_{d-1}\Theta(1) = |B|$ , we obtain that

$$N_1(t) \leq 2A_{d-1}\Theta(1)A_d\kappa_d \int_0^\infty \frac{dr r^{d-1}}{(1+r^2)^{\frac{d+1}{2}}} = |B| \cdot \|p_1^{(1)}\|_{L^1(\mathbb{R}^d)} = |B|,$$

so that by (5.4), we have

$$\frac{1}{\pi} \text{Per}(B) t N_2(t) = N_1(t) - \mathbb{H}_B^{(1)}(t) \leq |B| - \mathbb{H}_B^{(1)}(t).$$

Observe that (4.11) tells us that

$$\lim_{t \rightarrow 0+} \frac{|B| - \mathbb{H}_B^{(1)}(t)}{t \ln\left(\frac{1}{t}\right)} \leq \frac{1}{\pi} \text{Per}(B).$$

Hence, to prove (4.12), it suffices to show the following.

**Proposition 5.1.** *Consider the function  $N_2(t)$  defined in (5.5). Then,*

$$1 \leq \lim_{t \rightarrow 0+} \frac{N_2(t)}{\ln\left(\frac{1}{t}\right)}.$$

*Proof.* We recall the following basic inequality. For  $0 < x < 1$  and  $d \geq 2$ , we have

$$(5.6) \quad (1-x)^{\frac{d-1}{2}} \geq 1 - \sigma_d x,$$

where  $\sigma_d = \mathbb{1}_{\{2\}}(d) + \frac{(d-1)}{2} \cdot \mathbb{1}_{[3,\infty)}(d)$ . Based on the definition (5.5) of  $N_2(t)$ , we see that for any  $t < 2$  we have

$$(5.7) \quad \begin{aligned} N_2(t) &\geq \int_1^{2t^{-1}} \frac{dr r^d}{(1+r^2)^{\frac{d+1}{2}}} \left( 1 - \frac{t^2 r^2}{4} \right)^{\frac{d-1}{2}} = \int_1^{2t^{-1}} \frac{dr}{r} \left( 1 - \frac{t^2 r^2}{4} \right)^{\frac{d-1}{2}} - \Phi(t) \\ &\geq \int_1^{2t^{-1}} \frac{dr}{r} \left( 1 - \frac{\sigma_d}{8} t^2 r^2 \right) - \Phi(t) \geq \ln\left(\frac{1}{t}\right) - \frac{\sigma_d}{4} - \Phi(t), \end{aligned}$$

where to obtain the second inequality, we have used (5.6) and  $\Phi(t)$  has been defined by

$$\Phi(t) = \int_1^{2t^{-1}} dr \left( \frac{1}{r} - \frac{r^d}{(1+r^2)^{\frac{d+1}{2}}} \right) \left( 1 - \frac{t^2 r^2}{4} \right)^{\frac{d-1}{2}}.$$

Next, notice that  $\Phi(t)$  satisfies that

$$(5.8) \quad \Phi(t) \leq \int_1^\infty dr \left( \frac{1}{r} - \frac{r^d}{(1+r^2)^{\frac{d+1}{2}}} \right) < \infty.$$

Finally, by combining (5.7) with (5.8), we arrive at the desired result and this finishes the proof of Proposition 5.1.  $\square$

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